ON A MODEL FOR COMPUTING ROUND-OFF ERROR OF A SUM

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STAN-CS-70-156
MARCH 1970

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Technical Report No. 156

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Reproduction and research of this report was partially supported by Office of Naval Research, Contract N00014-67-A0112-0011; U.S. Atomic Energy Commission, Contract AT[04-3] 326 PA #18; National Science Foundation, Grant GP 9329 and Grant GJ 320; U.S. Army Research Office, Contract DAHC 04-67-0028; and National Institutes of Health, Grant GM 14789-02.

→ a sub 1, a sub 2, ..., a sub n, the report d...

Given real numbers $a_1, a_2, \ldots, a_n$, ~~we are interested in~~ the classic problem of the error in computing $S = \sum_1^n a_i$ when the sum is computed by $\tilde{S}_0 = \sum_1^n a_i^*$ where a_i^* is the nearest integer to a_i. We shall first study this error as a function of a Δ shift, i.e., when all numbers a_i are each shifted Δ and then rounded;

→ (a sub i)*

→ a sub i

2nd

$$(1) \qquad S - n\Delta = \sum_{i=1}^{n} (a_i - \Delta)$$

$$(2) \qquad \tilde{S}_\Delta - n\Delta = \sum_{i=1}^{n} (a_i - \Delta)^*$$

We will then let Δ become a random variable that can take on uniformly any value in the interval $-\frac{1}{2} \leq \Delta \leq +\frac{1}{2}$. Different choices of Δ give rise to different rounding errors $\tilde{S}_\Delta - S$ and the variance of the distribution of $\tilde{S}_\Delta - S$ can be used to measure the variability of the <u>rounding error due to the random selection of the origin of the real numbers</u> a_i <u>with respect to that of the computer</u>.

The cumulative error from (1) and (2) is

$$(3) \qquad \tilde{S}_\Delta - S = \sum_{i=1}^{n} [(a_i - \Delta)^* - (a_i - \Delta)]$$

Let f_i be the positive fractional part of a_i and let α_i be the largest integer not exceeding a_i, i.e.,

$$(4) \qquad a_i = \alpha_i + f_i$$

Denoting by r_i the error of the i^{th} term, we have

$$(5) \quad r_i = [(a_i-\Delta)^* - (a_i-\Delta)] = \begin{cases} 1-(f_i-\Delta) & \text{if } -\frac{1}{2} \le \Delta \le -\frac{1}{2} + f_i \\ -(f_i-\Delta) & \text{if } -\frac{1}{2} + f_i \le \Delta \le +\frac{1}{2} \end{cases}$$

To prove the above, we note that $f_i-\Delta = (a_i-\Delta) + \alpha_i$. If $-\frac{1}{2} \le f_i - \Delta \le +\frac{1}{2}$ then $(a_i-\Delta)$ is rounded to α_i. Hence $a_i-\Delta$ is rounded down if $-\frac{1}{2} + f_i \le \Delta$ otherwise rounded up.

Denoting expected value by E, we have by direct evaluation

$$(6) \qquad E(r_i) = \int_{-\frac{1}{2}}^{+\frac{1}{2}} r_i d\Delta = 0$$

Assume $f_i \le f_j$, then

$$E(r_i r_j) = \int_{-\frac{1}{2}}^{-\frac{1}{2}+f_i} r_i r_j d\Delta + \int_{-\frac{1}{2}+f_i}^{-\frac{1}{2}+f_j} r_i r_j d\Delta + \int_{-\frac{1}{2}+f_j}^{+\frac{1}{2}} r_i r_j d\Delta$$

$$= \int_{-\frac{1}{2}}^{+\frac{1}{2}} (f_i f_j - \Delta(f_i + f_j) + \Delta^2)\, d\Delta$$

$$+ \int_{-\frac{1}{2}}^{-\frac{1}{2}+f_i} [(1-f_i-f_j) + 2\Delta]\, d\Delta$$

$$+ \int_{-\frac{1}{2}+f_i}^{-\frac{1}{2}+f_j} [-f_i + \Delta)\, d\Delta$$

Performing indicated integration yields:

$$(7) \qquad E(r_i r_j) = \frac{1}{2}[|f_j-f_i|^2 - |f_j-f_i| + \frac{1}{6}]$$

which is one-half the 2^{nd} order Bernoulli Polynomial in $|f_j - f_i|$. For $f_j \leq f_i$ we also get (7). Note that the individual errors r_i and r_j are <u>not</u> independent of one another.

It now follows that

$$(9) \qquad E(\tilde{S}) = S$$

$$(10) \qquad E(\tilde{S}-S)^2 = E\left(\sum_{i=1}^{n} \sum_{j=1}^{n} r_i r_j\right) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[|f_i - f_j|^2 - |f_i - f_j| + \frac{1}{6}\right]$$

The usual value of variance, $E(\tilde{S}-S)^2 = n/12$, will result if we further assume f_i are independently drawn from uniform distributions on $[0 \leq f_i \leq 1]$.

<u>Theorem</u>: If the fractional parts of all a_i are equal to each other, then each term of (10) is maximum for $0 \leq f_i \leq 1$ and

$$(11) \qquad \text{Max } E(S-S)^2 = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\frac{1}{6}\right) = \frac{n^2}{12} \quad .$$

From (10) we have an interesting inequality, namely for all f_i

$$(12) \qquad V(f) = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \{|f_i - f_j|^2 - |f_i - f_j| + \frac{1}{6}\} \geq 0$$

This function is not convex even for $n=2$, since $f^{(1)} = (\frac{1}{2}, 0)$ and $f^{(2)} = (-\frac{1}{2}, 0)$ yields $V(f^0) = V(f^1) = \frac{1}{12} + \frac{1}{12} - \frac{1}{12} = \frac{1}{12}$ but

$V(\frac{f^1 + f^2}{2}) = V(0) = \frac{3}{12}$. There appears to be no obvious direct way to establish that $V(f) \geq 0$ for all $0 \leq f_i \leq 1$. Our development shows $V(f)$ to be a variance and this, of course, constitutes an indirect proof. We can replace (12) by a convex realization: Assume $f_i \geq f_{i+1}$ for all i, then the problem of finding Min $V(f)$ can be rewritten:

$$(13) \quad \text{Find Min } [V(f)] = \sum_{i<j}\sum (f_i - f_j)^2 + \frac{n^2}{12} - [(n-1)f_1 + (n-3)f_2 + (n-5)f_3 + \ldots + (n-2k+1)\, f_k + \ldots - (n-1)f_n]$$

subject to

$$(14) \quad f_1 \geq f_2 \ldots \geq f_n$$

$$(15) \quad 0 \leq f_i \leq 1$$

Formally (13), (14), (15), is a positive definite quadratic program. Fortunately, as we shall see this can be solved by classical calculus by ignoring inequalities (14) and (15).

<u>Theorem</u>: Equally spaced $f_i = (n - i)/n$, $(i = 1,\ldots,n)$ yields Min $V(f) = \frac{1}{12}$ independent of n, i.e., the variance of the sum in this case is minimum and is the same as the variance of the individual terms forming the sum.

Proof: Setting partials $= 0$ in (13) yields:

$$
(16)\quad \begin{cases}
2(n-1)f_1 \quad - 2f_2 \quad \cdots \quad - 2f_n = (n-1) \\
- 2f_1 \quad + 2(n-1)f_2 \quad \quad - 2f_n = (n-3) \\
- 2f_1 \quad - 2f_2 \ldots 2(n-1)f_{n-1} - 2f_n = -(n-3) \\
- 2f_1 \quad - 2f_2 \quad \quad 2(n-1)f_n = -(n-1)
\end{cases}
$$

Adding shows the equations to be dependent. Hence we may drop the last equation as redundant. Moreover, we can always translate the f_i so that the smallest f_i, namely $f_n = 0$

Re-adding yields:

$$2f_1 + 2f_2 + \ldots 2f_{n-1} + 0 = (n-1) \qquad , f_n = 0.$$

Adding this last equation to each of the others gives

$$2nf_i = (n - 2i + 1) + (n - 1) = 2(n - i)$$

$$(17) \qquad f_i = (n - i)/n$$

Evidently the conditions $0 \leq f_i \leq 1$ and $f_i \geq f_{i+1}$ are (by good luck) also satisfied so that (17) yields the minimum, namely

$$(18) \qquad \text{Min } V(f) = \frac{n^2}{12} - \frac{1}{2} \sum_{i=1}^{n} (n-2_i+1)f_i = \frac{1}{12} \ .$$